

# Casimir Energies: Temperature Dependence, Dispersion, and Anomalies

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## Abstract

Assuming the conventional Casimir setting with two thick parallel perfectly conducting plates of large extent with a homogeneous and isotropic medium between them, we discuss the physical meaning of the electromagnetic field energy  $W_{\text{disp}}$  when the intervening medium is weakly dispersive but nondissipative. The presence of dispersion means that the energy density contains terms of the form  $d[\omega\varepsilon(\omega)]/d\omega$  and  $d[\omega\mu(\omega)]/d\omega$ . We find that, as  $W_{\text{disp}}$  refers thermodynamically to a non-closed physical system, it is *not* to be identified with the internal thermodynamic energy  $U$  following from the free energy  $F$ , or the electromagnetic energy  $W$ , when the last-mentioned quantities are calculated without such dispersive derivatives. To arrive at this conclusion, we adopt a model in which the system is a capacitor, linked to an external self-inductance  $L$  such that stationary oscillations become possible. Therewith the model system becomes a non-closed one. As an introductory step, we review the meaning of the nondispersive energies,  $F$ ,  $U$ , and  $W$ . As a final topic, we consider an anomaly connected with local surface divergences encountered in Casimir energy calculations for higher spacetime dimensions,  $D > 4$ , and discuss briefly its dispersive generalization. This kind of application is essentially a generalization of the treatment of Alnes *et al.* [J. Phys. A: Math. Theor. **40**, F315 (2007)] to the case of a medium-filled cavity between two hyperplanes.

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## I. INTRODUCTION

For many years, after its discovery in 1948 [1] the Casimir effect was a theoretical curiosity, although it had evident applications to van der Waals forces [2] and models of hadrons [3]. The Casimir formula for the quantum vacuum force between conducting plates was generalized to dielectrics by Lifshitz in 1956 [4], and the 1973 experiment of Sabisky and Anderson [5], testing the Lifshitz prediction to a good accuracy, is well known.

But the renaissance in studies of the Casimir effect began in 1997 with the work of Lamoreaux [6]. He measured the Casimir force between a conducting plate and a spherical lens, which, through the proximity force approximation [7, 8, 9], agreed with expectations at something like the 5% level. (The accuracy of this measurement remains under some dispute, because various corrections, such as the effects of surface roughness, patch potentials, and finite conductivity, were not adequately taken into account.) In subsequent years, a variety of experiments were carried out, some of much greater accuracy and at considerably shorter distances, Refs. [10, 11, 12, 13, 14, 15, 16, 17, 18, 19], which have incorporated various corrections [20].

One reason why the Casimir effect has attracted so much attention in recent years is the question of the temperature correction to the attractive force between real metal surfaces. At large distances (some micrometers) the relative thermal correction achieves several tens of percent, but at these distances the force itself becomes weak, and the experimental technique is not sufficiently sensitive to give clear-cut results. At small distances, around 100 nanometers, the measurements are claimed to be of high accuracy, about 1%, but at such distances the thermal correction is relatively small. On the theoretical side, the process of extracting the temperature dependence was carried out with the prescription given in Ref. [21]. Because of inaccessibility of the effect to precise experiments, the issue was not reconsidered until the modern era, when Boström and Sernelius [22] recognized that this prescription could not be correct, and that necessarily the transverse electric reflection coefficient at zero frequency must vanish for metals. This led to a reduction by a factor of two in the prediction for the slope of the linear high-temperature behavior (which would only be visible in experiments carried out at several microns), but it would predict a new linear temperature term at low temperatures, resulting in a 15% correction to the result found by Lamoreaux. Lamoreaux believes that his experiment could not be in error to this extent [23]. Mostepanenko and collaborators have insisted that this behavior is inconsistent with thermodynamics (the Nernst heat theorem), because it would predict that the free energy has a term linear in  $T$  at low temperature. Such a behavior is predicted by the so-called modified ideal metal model and also, as advocated by these authors, when the Drude model is applied to the case of a metal with perfect crystal lattice without impurities, in which relaxation of conduction electrons is only due to scattering on thermal phonons [24, 25, 26]. Moreover, they assert that the precision Purdue experiments [18], performed at  $T = 300$  K rule out the large thermal corrections predicted by the use of the Drude model with lattice imperfections taken into account [27]. The first Purdue experiment was performed at distances larger than 260 nm. More exact repetitions of that experiment [28] have been performed, down to 160 nm. We and others have responded that real metals do not exhibit this thermodynamic inconsistency, and that most probably the experiments are not so accurate as claimed [27, 29, 30]. The situation is summarized in recent reviews [31, 32]. In particular, the lack of a thermodynamic inconsistency has been conclusively demonstrated [33, 34], by showing that the free energy for a Casimir system made from real metal plates

with impurities has a quadratic temperature dependence at low temperature. Further evidence for the validity of the notion of excluding the TE zero mode for metals comes from the recent work of Buenzli and Martin [35], corroborating earlier work by these authors and others [36, 37], who show from a microscopic viewpoint that the high-temperature behavior of the Casimir force is half that of an ideal metal, a rather conclusive demonstration that the TE zero mode is not present.<sup>1</sup>

Our purpose with the present paper is not to study the temperature corrections in further detail. The brief survey above indicates that there is a need of reconsidering the underpinnings of the Casimir theory in some detail. As an attempt to do this, we will consider the Casimir problem from an unconventional angle, emphasizing the role of dispersive media. Thus, in Sec. III we will show how the Casimir energy  $W_{\text{disp}}$  for a dispersive nondissipative medium, reflecting a *non-closed* physical system, is not to be identified with the internal thermodynamic energy  $U$ , or the electromagnetic energy  $W$ , when  $U$  and  $W$  are calculated as though dispersion were not present. In this regard a capacitor model of the system proves to be quite illuminating. Finally, in Sec. IV we examine another aspect of phenomenological electrodynamics, namely its generalization to higher spacetime dimensions,  $D > 4$ , both because the topic has some relationship to the dispersive theory discussed in Sec. III, and also because the higher-dimensional electrodynamical theory is a topic of general current interest.

## II. FREE ENERGY $F$ , INTERNAL ENERGY $U$ , AND ELECTROMAGNETIC ENERGY $W$

In order to fix the notation, and for reference purposes, we give in this section a brief survey of how the various energy concepts occur in Casimir theory. Assume the usual configuration, in which there are two thick infinitely large plates situated at  $z = 0$  and  $z = a$ , with a homogeneous and isotropic medium in between. We take this intervening medium to have permittivity  $\varepsilon$  and permeability  $\mu$ . In this section we take these material parameters to be constants. For simplicity we assume that the medium to the left ( $z < 0$ ), as well as that to the right ( $z > a$ ) are ideal (the so-called IM model), so that the TE and TM reflection coefficients  $r_{\text{TE}}$  and  $r_{\text{TM}}$  satisfy  $r_{\text{TE}}^2 = r_{\text{TM}}^2 = 1$  for all Matsubara frequencies  $m$ , including  $m = 0$  (the breakdown of this assumption for the  $r_{\text{TE}}$  coefficient at  $m = 0$  is the crux of the temperature controversy for real metals). Let  $n = \sqrt{\varepsilon\mu}$  be the refractive (temperature independent) index of the intervening medium,  $\beta = 1/T$ ,  $\zeta_m = 2\pi m/\beta$ , and  $\kappa^2 = k_{\perp}^2 + n^2\zeta_m^2$ . The free energy  $F$  per unit surface area can now be written

$$F = \frac{1}{\pi\beta} \sum'_{m=0} \int_{n\zeta_m}^{\infty} \kappa d\kappa \ln(1 - e^{-2\kappa a}), \quad (2.1)$$

for arbitrary  $T$ . The prime on the summation sign means that the  $m = 0$  term is counted with half-weight.

The internal energy per unit area  $U$  is now constructed from the thermodynamical formula

$$U = \frac{\partial(\beta F)}{\partial\beta}. \quad (2.2)$$

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<sup>1</sup> It could here be added, as a contrast, that Intravaia and Henkel have recently claimed that for metals with perfect crystal lattices the Lifshitz theory leads to violation of Nernst's theorem [38].

From Eq. (2.1) it is apparent that  $\beta$  appears only in the lower limit of the integral in the expression for  $\beta F$ . Since  $\partial\zeta_m/\partial\beta = -2\pi m/\beta^2$ , we get

$$U = \frac{4\pi n^2}{\beta^3} \sum_{m=0}^{\infty} m^2 \ln(1 - e^{-\alpha m}), \quad (2.3)$$

where

$$\alpha = \frac{4\pi na}{\beta} = 4\pi naT. \quad (2.4)$$

The  $m = 0$  term does not contribute. One way of processing the expression (2.3) is to expand the logarithm,

$$m^2 \ln(1 - e^{-\alpha m}) = - \sum_{k=1}^{\infty} \frac{1}{k} m^2 e^{-\alpha k m}, \quad (2.5)$$

and then sum over  $m$ , whereby we get

$$U = -\pi n^2 T^3 \sum_{m=1}^{\infty} \frac{1}{m} \frac{\coth(2\pi n m a T)}{\sinh^2(2\pi n m a T)}. \quad (2.6)$$

When  $n = 1$ , this agrees with Eq. (18) of Ref. [40]. (Cf. also the discussion of energy and free energy in Ref. [41].) The expansion (2.6) is most convenient at high temperatures,  $aT \gg 1$ . By including only the  $m = 1$  term, one gets

$$U = -4\pi n^2 T^3 e^{-4\pi naT}, \quad aT \gg 1. \quad (2.7)$$

It is apparent that  $U \rightarrow 0$  when  $T \rightarrow \infty$ . This is as we should expect physically: The Casimir energy measures the change in energy induced by the boundaries, and these constraints decrease in importance when the classical thermal energy becomes high.

To get a convenient expression at low  $T$  one may perform a Poisson resummation, along the same lines as discussed in Ref. [21]. Define the quantity  $b(m)$ ,

$$b(m) = m^2 \ln(1 - e^{-\alpha|m|}), \quad (2.8)$$

along with its Fourier transform  $c(q)$ ,

$$c(q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} b(x) e^{-iqx} dx. \quad (2.9)$$

Then, according to the Poisson formula,

$$\begin{aligned} \sum_{m=-\infty}^{\infty} b(m) &= 2\pi \sum_{m=-\infty}^{\infty} c(2\pi m) = 2 \int_0^{\infty} x^2 \ln(1 - e^{-\alpha x}) dx \\ &\quad + 4 \sum_{m=1}^{\infty} \int_0^{\infty} x^2 \cos(2\pi m x) \ln(1 - e^{-\alpha x}) dx. \end{aligned} \quad (2.10)$$

The various terms can be evaluated analytically. The following formulas are here useful (Ref. [42], sec. 3.951), assuming  $b > 0$ ,

$$\int_0^{\infty} \frac{x^{2m} \sin bx}{e^x - 1} dx = (-1)^m \frac{\partial^{2m}}{\partial b^{2m}} \left[ \frac{\pi}{2} \coth \pi b - \frac{1}{2b} \right], \quad (2.11a)$$

$$\int_0^{\infty} \frac{x^{2m+1} \cos bx}{e^x - 1} dx = (-1)^m \frac{\partial^{2m+1}}{\partial b^{2m+1}} \left[ \frac{\pi}{2} \coth \pi b - \frac{1}{2b} \right]. \quad (2.11b)$$

The calculation gives, for arbitrary  $T$ ,

$$U = 2\pi n^2 T^3 \left[ -\frac{\pi}{1440(naT)^3} + \frac{naT}{\pi^3} \sum_{m=1}^{\infty} \frac{1}{m^4} \left\{ -3 + \frac{\pi m}{2naT} \coth \frac{\pi m}{2naT} + \frac{\left(\frac{\pi m}{2naT}\right)^2}{\sinh^2\left(\frac{\pi m}{2naT}\right)} \left[ 1 + \frac{\pi m}{2naT} \coth \frac{\pi m}{2naT} \right] \right\} \right]. \quad (2.12)$$

It is of interest to consider the limit of low dimensionless temperatures,

$$U = -\frac{\pi^2}{720na^3} \left[ 1 - 720 \left( \frac{naT}{\pi} \right)^3 \zeta(3) + 48(naT)^4 \right], \quad aT \ll 1. \quad (2.13)$$

Again, this agrees with the low-temperature expression obtained earlier, for instance in Ref. [43], when  $n = 1$ . It is to be noted that  $U$ , as well as the corresponding low-temperature expression for  $F$ ,

$$F = -\frac{\pi^2}{720na^3} \left[ 1 + 360 \left( \frac{naT}{\pi} \right)^3 \zeta(3) - (2naT)^4 \right], \quad (2.14)$$

contain a term that is independent of  $a$ , which means that this term does not contribute to the force between the plates.

The third kind of energy that we shall consider is the electromagnetic energy  $W$ , still taken per unit surface area. As above, we take the medium to be nondispersive. We start from the energy density,

$$w = \frac{1}{2} \varepsilon (E_z^2 + E_{\perp}^2) + \frac{1}{2} \mu (H_z^2 + H_{\perp}^2), \quad (2.15)$$

so that, per unit area,  $W = wa$ . Quantum mechanically, the product  $E_z^2(\mathbf{r})$  is to be replaced by the expectation value  $\langle E_z(\mathbf{r}) E_z(\mathbf{r}') \rangle$  in the limit when  $\mathbf{r}' \rightarrow \mathbf{r}$ . Similarly for the other components. We assume first that  $T = 0$ . According to the fluctuation-dissipation theorem in Fourier space we have

$$i \langle E_i(\mathbf{r}) E_k(\mathbf{r}') \rangle_{\omega} = \text{Im} \Gamma_{ik}(\mathbf{r}, \mathbf{r}'; \omega), \quad (2.16a)$$

$$i \langle H_i(\mathbf{r}) H_k(\mathbf{r}') \rangle_{\omega} = \frac{1}{\mu^2 \omega^2} \text{curl}_{ij} \text{curl}'_{kl} \text{Im} \Gamma_{jl}(\mathbf{r}, \mathbf{r}'; \omega), \quad (2.16b)$$

where  $\text{curl}_{ik} \equiv \epsilon_{ijk} \partial_j$ ,  $\epsilon_{ijk}$  being the Levi-Civita symbol. Further,  $\mathbf{\Gamma}$  is the Green's function as defined by Schwinger et al. [21], in terms of a polarization source  $\mathbf{P}$ ,

$$\mathbf{E}(x) = \int d^4 x' \mathbf{\Gamma}(x, x') \cdot \mathbf{P}(x'), \quad (2.17)$$

with

$$\mathbf{\Gamma}(x, x') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} \mathbf{\Gamma}(\mathbf{r}, \mathbf{r}'; \omega), \quad (2.18)$$

and  $\tau = t - t'$ . Introducing a transverse Fourier transform,

$$\mathbf{\Gamma}(\mathbf{r}, \mathbf{r}'; \omega) = \int \frac{d^2 k_{\perp}}{(2\pi)^2} e^{i\mathbf{k}_{\perp} \cdot (\mathbf{r} - \mathbf{r}')} \mathbf{g}^E(z, z'; \mathbf{k}_{\perp}, \omega), \quad (2.19)$$

we can write

$$g_{xx}^E = -\frac{\kappa}{\varepsilon} \frac{1}{d} \cosh \kappa(z - z'), \quad (2.20a)$$

$$g_{yy}^E = \frac{\mu\omega^2}{\kappa} \frac{1}{d} \cosh \kappa(z - z'), \quad (2.20b)$$

$$g_{zz}^E = \frac{k_{\perp}^2}{\kappa\varepsilon} \frac{1}{d} \cosh \kappa(z - z'), \quad (2.20c)$$

where

$$d = e^{2\kappa a} - 1, \quad \kappa^2 = k_{\perp}^2 - n^2\omega^2. \quad (2.21)$$

(Details are given in Ref. [44].) (Note that the notation is slightly different than that given in Ref. [45].)

Defining the Fourier components  $\langle \dots \rangle_{\omega\mathbf{k}}$  of the energy density according to

$$w = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^2 k_{\perp}}{(2\pi)^2} [\varepsilon \langle E^2 \rangle_{\omega\mathbf{k}} + \mu \langle H^2 \rangle_{\omega\mathbf{k}}], \quad (2.22)$$

we first obtain for the electric part, letting  $z' \rightarrow z$ ,

$$\frac{1}{2} \varepsilon \langle E^2 \rangle_{\omega\mathbf{k}} = \frac{1}{i} \frac{\varepsilon}{2} (g_{xx}^E + g_{yy}^E + g_{zz}^E) = \frac{n^2\omega^2}{i\kappa} \frac{1}{d}. \quad (2.23)$$

Then defining the magnetic counterpart  $g_{ik}^H$  to the electric part  $g_{ik}^E$  according to

$$g_{ik}^H = \frac{1}{\omega^2} \text{curl}_{il} \text{curl}'_{km} g_{lm}^E, \quad (2.24)$$

we obtain by an analogous calculation, in the limit when  $z' \rightarrow z$ ,

$$\frac{1}{2} \mu \langle H^2 \rangle_{\omega\mathbf{k}} = \frac{1}{i} \frac{1}{2\mu} (g_{xx}^H + g_{yy}^H + g_{zz}^H) = \frac{n^2\omega^2}{i\kappa} \frac{1}{d}. \quad (2.25)$$

The electric and magnetic contributions to the energy are equal, as we would expect. Adding the expressions (2.23) and (2.25) and multiplying with  $a$  we obtain, at zero temperature,

$$W = -\frac{n^2 a}{\pi^2} \int_0^{\infty} d\zeta \zeta^2 \int_0^{\infty} \frac{k_{\perp} dk_{\perp}}{\kappa d}, \quad (2.26)$$

where a frequency rotation  $\omega \rightarrow i\zeta$  has been performed. This expression can be further processed by introducing new coordinates  $X = k_{\perp} = \kappa \cos \theta$ ,  $Y = n\zeta = \kappa \sin \theta$ , with  $\kappa = \sqrt{k_{\perp}^2 + n^2\zeta^2}$ . We get

$$W = -\frac{1}{48\pi^2 n a^3} \int_0^{\infty} \frac{z^3 dz}{e^z - 1} = -\frac{\pi^2}{720 n a^3}, \quad (2.27)$$

in accordance with Eqs. (2.13) and (2.14).

At arbitrary temperature we get

$$W = -8\pi n^2 a T^3 \sum_{m=1}^{\infty} m^2 \int_0^{\infty} \frac{k_{\perp} dk_{\perp}}{\kappa d}, \quad (2.28)$$

with  $\kappa = \sqrt{k_{\perp}^2 + (2\pi n m T)^2}$ . Alternatively, we may write

$$W = -4\pi n^2 T^3 \sum_{m=1}^{\infty} m^2 \int_{\alpha m}^{\infty} \frac{dz}{e^z - 1}, \quad (2.29)$$

where  $\alpha = 4\pi n a T$  as before.

At high temperature,  $aT \gg 1$ , it is easy to check that  $W$  agrees with  $U$  calculated previously. We approximate the integral in Eq. (2.29) by  $\int_{\alpha m}^{\infty} e^{-z} dz = e^{-\alpha m}$ , and so get

$$W = -4\pi n^2 T^3 \sum_{m=1}^{\infty} m^2 e^{-\alpha m} \rightarrow -4\pi n^2 T^3 e^{-4\pi n a T} \quad (2.30)$$

when  $m = 1$ , in agreement with Eq. (2.7).

We shall not delve further into a detailed study of the equality between  $W$  and  $U$  in the case of arbitrary  $T$ . The equality should be clear on physical grounds, since we are dealing with a closed thermodynamical system.

After having given this survey, we have the necessary reference background for studying the dispersive regime.

### III. ON THE DISPERSIVE CASE, NEGLECTING DISSIPATION

As mentioned, our main focus will be on the dispersive case. Assume first that the medium in the region  $0 < z < a$  is both electrically and magnetically frequency dispersive,  $\varepsilon = \varepsilon(\omega)$ ,  $\mu = \mu(\omega)$ . The walls are taken to be perfectly conducting, as before. The total energy density  $w_{\text{disp}}$  is known to be [46, 47]

$$w_{\text{disp}} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^2 k_{\perp}}{(2\pi)^2} \left[ \frac{d(\varepsilon\omega)}{d\omega} \langle E^2 \rangle_{\omega \mathbf{k}} + \frac{d(\mu\omega)}{d\omega} \langle H^2 \rangle_{\omega \mathbf{k}} \right]. \quad (3.1)$$

We can write this as a sum of two parts  $w_I$  and  $w_{II}$ , where  $w_I$  is the same expression as in Eq. (2.22) with  $\varepsilon \rightarrow \varepsilon(\omega)$ ,  $\mu \rightarrow \mu(\omega)$ , and where

$$w_{II} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \omega \int \frac{d^2 k_{\perp}}{(2\pi)^2} \left[ \frac{d\varepsilon}{d\omega} \langle E^2 \rangle_{\omega \mathbf{k}} + \frac{d\mu}{d\omega} \langle H^2 \rangle_{\omega \mathbf{k}} \right]. \quad (3.2)$$

Correspondingly, for the surface densities,  $W_{\text{disp}} = W_I + W_{II}$ .

The first property to be noted in connection with Eq. (3.1) is that it is derived under the assumption of *negligible dissipation*. Some dissipation is always present—this being a consequence of Kramers-Kronig's relations—but it is a legitimate approximation to neglect it except in the neighborhood of eigenfrequencies in the cavity. This assumption means that the relaxation frequency in the dispersion relation can be set equal to zero, and we may adopt the usual dispersion relation for a dielectric, for simplicity taking it hereafter to be nonmagnetic,

$$\varepsilon(\omega) = 1 + \frac{\bar{\varepsilon} - 1}{1 - \omega^2/\omega_0^2}, \quad \mu = 1. \quad (3.3)$$

In the case of a general dissipative medium, neither the energy nor the stress tensor are derivable in terms of permittivity/permeability alone, and therefore cannot be given in a

general form using macroscopic methods. (This point is discussed in detail by Ginzburg [48].)

Second, it is clear that the expression (3.1) is not intimately related to the Casimir effect as such. It is more natural to consider the problem as belonging to classical electrodynamics, namely a system of two conducting plates between which there are stationary electromagnetic oscillations. The expression (3.1) is actually obtained from the energy balance equation

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{H} \cdot \dot{\mathbf{B}} = 0. \quad (3.4)$$

(See, for example, Eq. (7.5) in Ref. [47].) In order to accumulate electromagnetic energy, one has to consider oscillations that are not purely monochromatic, but distributed within a band of frequencies around each eigenfrequency. In this way external agencies, outside of the plates, are called for. It is natural here to regard the system to be a capacitor, linked to an external appropriately adjusted self-inductance  $L$  such that stationary oscillations become possible (external resistances are forbidden since we omit dissipation). That means, the plates with the intervening medium is thermodynamically a *non-closed* system. From this we can draw the important conclusion that the full dispersive energy  $W_{\text{disp}}$  is not to be identified with the thermodynamical energy  $W = U$  calculated earlier. The laws of thermodynamics are applicable to *closed* systems only.

The mentioned model of a classical electromagnetic non-dissipative circuit is studied in Ref. [46]. It is instructive to consider the salient features of the argument also here:

Let the charges  $Q$  be supplied and withdrawn from the plates with frequency  $\omega$ . The self-inductance of the circuit is  $L$ , as mentioned, and the electromotive force we call  $\mathcal{E}$ . The potential  $\phi$  across the plates is determined from the equation

$$\phi = \mathcal{E} - LJ, \quad (3.5)$$

where  $J = \dot{Q}$ . The frequency of the circuit is

$$\omega = 1/\sqrt{LC(\omega)}, \quad (3.6)$$

where the capacitance  $C(\omega)$  of the capacitor is determined by  $\phi = Q/C(\omega)$ . By considering almost monochromatic quantities [the same kind of argument that led to Eq. (3.1)], we get, when taking the average over a period,

$$\overline{\mathcal{E}J} = \frac{d}{dt} \left\{ \frac{1}{2} L \overline{J^2} + \frac{1}{2} \frac{d(\omega C)}{d\omega} \overline{\phi^2} \right\}. \quad (3.7)$$

The expression between brackets is the circuit energy. From  $J = -i\omega Q$  and Eq. (3.6) we get  $\frac{1}{2} L \overline{J^2} = \frac{1}{2} C \overline{\phi^2}$  and so the circuit energy may be written

$$\overline{W}_{\text{circ}} = \frac{1}{2\omega} \frac{d(\omega^2 C)}{d\omega} \overline{\phi^2}. \quad (3.8)$$

This expression, because of the derivative with respect to  $\omega$ , is seen to be related to Eq. (3.1).

Now consider a small adiabatic displacement of the capacitor plates. As  $\overline{W}_{\text{circ}}/\omega$  is an adiabatic invariant,

$$\delta \overline{W}_{\text{circ}} = \overline{W}_{\text{circ}} \delta \omega / \omega. \quad (3.9)$$

By means of Eq. (3.6),

$$\frac{\delta \omega}{\omega} = -\frac{1}{2} \frac{\delta C}{C}. \quad (3.10)$$



The change in  $C$  consists of two parts,

$$\delta C = (\delta C)_{\text{st}} + \frac{dC}{d\omega} \delta\omega, \quad (3.11)$$

where the first term is the static part and the second term depends on the frequency change. From Eqs. (3.10) and (3.11),

$$\delta C_{\text{st}} = -\frac{1}{\omega^2} \frac{d(\omega^2 C)}{d\omega} \delta\omega. \quad (3.12)$$

When Eq. (3.8) is substituted in Eq. (3.9) and (3.12) is used,  $dC/d\omega$  disappears, and we get

$$\delta \overline{W}_{\text{circ}} = -\frac{1}{2} \overline{\phi^2} (\delta C)_{\text{st}} = -\frac{1}{2} \frac{\overline{Q^2}}{C^2} (\delta C)_{\text{st}}. \quad (3.13)$$

This is the same expression as one obtains by taking the variation of the average of the energy  $Q^2/2C$  of a thermally insulated capacitor. It means that when dispersion is present, the electromagnetic stress tensor contains no derivatives with respect to the frequency. The argument is general, and is not critically dependent on our choice of a capacitor model.

When applied to our case, we can thus conclude as follows:

1. The dispersive energy  $W_{\text{disp}}$  whose density is given in Eq. (3.1) refers thermodynamically to a non-closed system, and is therefore not to be identified with the internal energy  $U$  calculated in Sec. II starting from the free energy  $F$ , or the electromagnetic energy  $W$ , in the nondispersive case. As was demonstrated, when  $\varepsilon$  and  $\mu$  are constants,  $W = U$ . We are still to use the same expressions for  $W$  and  $U$  when the permittivity and permeability depend on frequency.
2. As for the electromagnetic stress tensor, the derivatives with respect to  $\omega$  are not to be included. That is, the electromagnetic force can be calculated from Eq. (2.22) with  $\varepsilon \rightarrow \varepsilon(\omega)$ ,  $\mu \rightarrow \mu(\omega)$ .

It may finally be noted that by inserting the simple form (3.3) for  $\varepsilon(\omega)$  for a dielectric, we obtain for the dispersive correction  $W_{II} = a w_{II}$  a divergent expression,

$$W_{II} = \frac{2a(\bar{\varepsilon} - 1)}{\omega_0^2} \int_0^\infty \frac{d\omega}{2\pi} \frac{\omega^2}{(1 - \omega^2/\omega_0^2)^2} \int \frac{d^2 k_\perp}{(2\pi)^2} \langle E^2 \rangle_{\omega \mathbf{k}}; \quad (3.14)$$

cf. Eq. (3.2).

Another way to see that the dispersive medium should be treated without the frequency derivative of the permittivity is to recognize that the Casimir energy may be derived by a variation expression

$$\frac{\delta E}{A} = \frac{i}{2} \int \frac{d\omega}{2\pi} \frac{d^2 k_\perp}{(2\pi)^2} dz \delta\varepsilon(z) g_{kk}^E(z, z, \mathbf{k}_\perp, \omega), \quad (3.15)$$

which is Eq. (2.26) of Ref. [21]. This starting point is equivalent to the variational argument recounted in this section.

#### IV. DISCUSSION ON AN ANOMALY IN THE CASIMIR ENERGY FOR HIGHER DIMENSIONS

The electromagnetic theory in a continuous medium has in general a rich structure. Most notable is the fact that (within the commonly accepted Minkowski theory) the spatial photon momentum is equal to  $\mathbf{k} = n\omega\hat{\mathbf{k}}$ , implying that the photon four-momentum becomes spacelike,  $k^\mu k_\mu = (n^2 - 1)\omega^2 > 0$  (we make use of the metric  $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ , and assume  $n$  constant to begin with). Accordingly, there are inertial systems in which the photon energy becomes negative. A striking example of this sort is provided by the Čerenkov effect: In the inertial system where the emitting particle is initially at rest, the recoil kinetic energy of the particle is necessarily positive. Thus, in order to preserve energy conservation, the energy of the emitted photon has to be negative. Another example of a related sort is provided by the so-called anomalous Doppler effect, occurring when a quantum particle detector moves superluminally in the medium. Thus Ginzburg and Frolov [49] studied such kinds of particle detectors and showed how the excitation of a detector uniformly accelerated in a vacuum with the associated emission of radiation is actually similar to the radiation occurring in the region of the anomalous Doppler effect when the detector is moving superluminally with constant velocity in the medium. See also the discussion in Ginzburg's book [48]. Situations of these kinds were discussed also by Brevik and Kolbenstvedt, in the case of constant velocity [50] and for constant acceleration [51].

We shall round off our paper not by considering the above-mentioned effects any further, but instead another effect that has also a bearing on medium electrodynamics, namely the anomaly that turns up in the case of higher spacetime dimensions,  $D > 4$ . The anomaly reflects the breaking of conformal symmetry. We do this because the topic has some relationship to that considered in Sec. III, and also because it has attracted interest recently in the case of a vacuum field. A generalization to the medium case thus appears natural. Higher dimensions, in the context of Casimir theory, were considered long ago by Ambjørn and Wolfram [52], but anomalies of the type considered below were not studied until recently by Alnes *et al.* [39]; cf. also Refs. [53, 54].

Let us assume, then, that there are two parallel hyperplanes with separation  $a$ , the region  $0 < z < a$  being filled with an isotropic medium of refractive index  $n = \sqrt{\varepsilon\mu}$ . The walls are assumed perfectly conducting, as before. The anomaly we wish to consider is present also in the case of zero temperature, so we shall assume  $T = 0$  in the following.

The appropriate electromagnetic energy-momentum tensor is the Minkowski expression, called  $S_{\mu\nu}^M$ ,

$$S_{\mu\nu}^M = F_{\mu\alpha}H_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}H^{\alpha\beta}; \quad (4.1)$$

cf., for instance, Refs. [55, 56, 57]. Here  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  with  $\mu, \nu = 0, 1, 2, \dots, D-1 = d$  is the field tensor, whereas  $F^{0k} = E_k$  with  $k = 1, 2, \dots, d$  are the components of the  $d$ -dimensional electric field vector  $\mathbf{E}$ . The magnetic induction ( $\mathbf{B}$  in the three-dimensional case) does not constitute a vector in the higher-dimensional case, but is given by the  $d(d-1)/2$  components of the antisymmetric spatial tensor  $F_{ik}$ . Analogously, the second  $D$ -dimensional tensor  $H_{\mu\nu}$  occurring in Eq. (4.1) is given by the vector components  $H^{0k} = D_k$ ,  $\mathbf{D}$  being the  $d$ -dimensional induction vector, and by the  $d(d-1)/2$  components of the spatial magnetic field tensor  $H_{ik}$  ( $\mathbf{H}$  in the three-dimensional case). In analogy with three-dimensional theory, we assume constitutive relations in the form  $H^{0k} = \varepsilon F^{0k}$  and  $F_{ik} = \mu H_{ik}$  also when  $D > 4$ .

Turning now to physical quantities, it is convenient to start with the surface pressure  $P$

on the hyperplane  $z = 0$ . We observe that the usual expression for  $P$  (cf. Eq. (2.1)) can easily be generalized to the case of  $d = D - 1$  spatial dimensions. Taking into account that there are  $(D - 2)$  physical degrees of freedom in the field in the cavity, we have

$$F = (D - 2) \int_0^\infty \frac{d\zeta}{2\pi} \int \frac{d^{d-1}k_\perp}{(2\pi)^{d-1}} \ln(1 - e^{-2\kappa a}), \quad (4.2)$$

where

$$\kappa^2 = k_\perp^2 + n^2 \zeta^2, \quad k_\perp^2 \equiv k_x^2 + k_y^2 + \dots + k_{D-2}^2. \quad (4.3)$$

The volume element in momentum space is  $d^{d-1}k_\perp = \Omega_{d-2} k_\perp^{d-2} dk_\perp$ , where the solid angle is determined by  $\Omega_{d-1} = 2\pi^{d/2} [\Gamma(d/2)]^{-1}$ . The pressure  $P = -\partial F / \partial a$  can now be written

$$P = -\frac{2(D-2)}{(2\pi)^d} \Omega_{d-2} \int_0^\infty d\zeta \int \frac{\kappa k_\perp^{d-2} dk_\perp}{e^{2\kappa a} - 1}. \quad (4.4)$$

The double integral over  $\zeta$  and  $k_\perp$  can be further processed by introducing polar coordinates, [21]. Again, we introduce  $X = \kappa \cos \theta = k_\perp$ ,  $Y = \kappa \sin \theta = n\zeta$ , satisfying  $X^2 + Y^2 = \kappa^2$ . The area element in the  $XY$  plane is  $\kappa d\kappa d\theta = n dk_\perp d\zeta$ . The integral therewith becomes

$$\int_0^\infty d\zeta \int \frac{\kappa k_\perp^{d-2} dk_\perp}{e^{2\kappa a} - 1} = \frac{1}{n} \int_0^{\pi/2} \cos^{d-2} \theta d\theta \int_0^\infty \frac{\kappa^d d\kappa}{e^{2\kappa a} - 1}. \quad (4.5)$$

We now make use of known integral formulas and insert the expression for  $\Omega_{d-2}$ , to get

$$P = -\frac{(D-2)(D-1)}{n} \frac{\Gamma(D/2)\zeta(D)}{(4\pi)^{D/2} a^D}. \quad (4.6)$$

It ought to be emphasized that this expression was obtained without any regularization procedure. The presence of the medium is seen here to turn up through the factor  $n$  in the denominator. If  $n = 1$ , including the case of a vacuum as well as the case of a “relativistic” medium satisfying  $\varepsilon = 1/\mu$ , the expression reduces to that derived earlier [52]. This result parallels that obtained in the  $T = 0$  parts of the energy, cf. Eqs. (2.13) and (2.14). It is nearly identical to the result found in Ref. [45] for the scalar case in  $D$  dimensions [Eq. (2.35) there], differing only in the factor  $(D-2)/n$ .

The electromagnetic field energy density  $w$  in the cavity is a more delicate quantity. The natural way to calculate  $w$  is via the energy-momentum tensor. This procedure - carried out by Alnes *et al.* in the case of a vacuum cavity [39, 54] - led in the case of metallic boundary conditions to the result

$$w = -\frac{(D-2)\Gamma(D/2)}{(4\pi)^{D/2} a^D} \left[ \zeta(D) + \left( \frac{D}{2} - 2 \right) f_D \left( \frac{z}{a} \right) \right] \equiv w_1 + w_2, \quad (4.7)$$

where

$$f_D \left( \frac{z}{a} \right) = \zeta_H \left( D, \frac{z}{a} \right) + \zeta_H \left( D, 1 - \frac{z}{a} \right), \quad (4.8)$$

$\zeta_H$  being the Hurwitz zeta function. Note that the first term yields the pressure (4.6),

$$-\frac{\partial}{\partial a} a w_1 = P, \quad (4.9)$$

so that the second term in the energy density,  $w_2$ , which diverges like  $z^{-D}$  close to the surface when  $D > 4$ , does not contribute to the force between the plates. It can explicitly be seen that written in physical variables this term is independent of the separation between the plates and hence does not contribute to the force. This anomaly can actually be seen to manifest itself in another way if we go back to the expression (4.1) for the energy-momentum tensor: its trace  $S_\mu^{M\mu}$  is *nonvanishing* when  $D > 4$ . Physically, as emphasized in Ref. [39], the divergent self energy of a single surface is related to the lack of conformal invariance of the electromagnetic Lagrangian for  $D > 4$ . All of this is exactly as seen in Ref. [45], Chap. 11, for the scalar field.

It turns out that the anomaly can be regularized away by subtracting off the self energy for both plates. Then, the second term in Eq. (4.7) is absent, and only the first, finite, terms in  $w$  remains.

As mentioned, these calculations of  $w$  were made for the case of a vacuum cavity. The result was found via a combination of dimensional and zeta function regularizations [58]. The result could be recalculated for a medium cavity, but such a detailed calculation is hardly justified in view of the simple occurrence of  $n$  in the expression (4.6). In fact, since in physical units,  $w = \hbar c/a^D$  times a function of  $D$ , it is clear (for example, Ref. [47], Eq. (36.12)) all we have to do to insert a uniform medium between the plates is replace  $c$  by  $c/n$ ; this shows that the same factor  $n$  will appear in the denominator of the expression for  $w$  as it did in  $W$  or  $P$ . Thus, after regularization, we obtain the relationship

$$P = (D - 1)w_1, \quad (4.10)$$

which is the same connection as for a vacuum.

Finally, we consider an alternative method for obtaining the energy  $W$  that avoids the field theoretical approach above, and instead starts by considering the individual photon momenta directly. The photon momentum in the medium is  $\sqrt{k_\perp^2 + \pi^2 m^2/a^2}$ , and the photon energy is obtained by dividing this expression by  $n$ , assuming that  $n = \text{constant}$ . Thus we have, still at  $T = 0$ ,

$$W = \frac{1}{n} \sum_{m=1}^{\infty} \int \frac{d^{d-1} k_\perp}{(2\pi)^{d-1}} \sqrt{k_\perp^2 + \pi^2 m^2/a^2}. \quad (4.11)$$

In order to extract a finite expression we have to regularize in some way, for instance by using an exponential cutoff. The important point in our context is however that the integral in Eq. (4.11), and the sum, are just the same as in a vacuum field. Thus the influence of the medium turns up only in the prefactor  $1/n$ , in accordance with what was found above.

Can this theory be generalized to the dispersive case? Such a performance is not quite straightforward, in view of the complicated form (3.1) for the dispersive energy density. Some insight can however be obtained from the following argument. As noted in Sec. III, the electromagnetic stress tensor does not contain derivatives with respect to the frequency. Thus, the momentum flux density has the same form as in a nondispersive medium. We may assume therefore that the photon wave vector is equal to  $n(\omega)\omega$ :

$$\sqrt{k_\perp^2 + \pi^2 m^2/a^2} = n(\omega)\omega. \quad (4.12)$$

When the wave vector is given, this equation can be solved (numerically) for  $n(\omega)$  and  $\omega$ . Inverting, we find  $n$  as a function of  $\sqrt{k_\perp^2 + \pi^2 m^2/a^2}$ . Accordingly, we can write the energy

as

$$W = \sum_{m=1}^{\infty} \int \frac{d^{d-1} k_{\perp}}{(2\pi)^{d-1}} \frac{\sqrt{k_{\perp}^2 + \pi^2 m^2/a^2}}{n(\sqrt{k_{\perp}^2 + \pi^2 m^2/a^2})}. \quad (4.13)$$

For very high wave numbers,  $n \rightarrow 1$ , and the integral reduces for these frequencies to its vacuum counterpart. One should bear in mind that the expression (4.13) holds only in an approximate sense, as we have ignored the accumulation of energy during the slow building up of the electromagnetic field.

## V. CONCLUSIONS

Some care ought to be taken when dealing with dispersive and dissipative media. In the general case of arbitrary dispersion (which implies necessarily dissipation also), the electromagnetic energy cannot be rationally defined as a thermodynamic quantity at all. If dispersion is weak, making it possible to ignore the accompanying dissipation, it is meaningful to define the electric energy such that it contains terms of the type  $d[\omega\varepsilon(\omega)]/d\omega$ , and similarly for the magnetic field. In such a case, as we have seen, one should distinguish the electromagnetic energy  $W_{\text{disp}}$  from the thermodynamic energies calculated for nondispersive media and used without the derivatives on  $\varepsilon$  and  $\mu$  in the dispersive case where  $n = n(\omega)$ . The main reason for the difference is that in the dispersive case we are dealing with a non-closed physical system.

The electrodynamic theory of media, especially when dispersion is included, has a rich structure. As an example of this, we showed in Sec. IV the anomaly turning up when  $D > 4$ , which is especially interesting when dispersion is present. This shows the interplay between local surface energy divergences and the breaking of conformal symmetry. The clarity brought to bear by the above analysis will now allow us to understand more fully the dispersive case, and to some extent also the questions connected with temperature problems. Moreover, we hope to have contributed to the understanding of surface energies and their significance.

Finally, we make the following comment. Our electromagnetic formalism in this paper has been the conventional one, whereby the basis for calculating stresses on matter is the Abraham-Minkowski stress tensor (cf., for instance, Ref. [56]). Now, in a recent paper Raabe and Welsch [59] have developed a somewhat unconventional theory for electromagnetic fields in a medium based upon the Lorentz force, from which they derive a stress tensor different from the Abraham-Minkowski form. The Casimir effect was chosen by these authors as the physical phenomenon to which they applied their proposed theory. We merely mention this novel formulation here; it would lead us too far from our main purpose to make a detailed scrutiny of this rather complicated formulation. A comment on some consequences of the altered stress tensor in practical applications is under preparation [60].

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